Optimization Procedure to Correct Stiffness and Flexibility Matrices Using Vibration Tests

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Nomenclature

F	= Lagrange function for the flexibility matrix
f	= weighted norm of the errors between the given and the optimal flexibility matrix
G	= Lagrange function for the stiffness matrix
g	= weighted norm of the errors between the given and the optimal stiffness matrix
I	= unity matrix
K	= given stiffness matrix
M	= mass matrix
N	$= M^{1/2}$
n_{ij}	= ij element of N
$\bar{n}_{ij}$	= ij element of N^{-1}
p	$= Nq$
q	= general-coordinates vector
T	= measured mode shape
$\tilde{T}_i$	= ith measured mode shape
T_i	= normalized $\tilde{T}_i$
$[\cdot]^t$	= transpose of $[\cdot]$
W	= optimal flexibility matrix
w_{ij}	= ij element of W
X	= orthogonal mode shape matrix
x_{ij}	= ij element of X
Y	= optimal stiffness matrix
y_{ij}	= ij element of Y
β_y, β_w	= matrices of Lagrange multipliers
$\beta_{y,ij}, \beta_{w,ij}$	= ij element of β_y and β_w, respectively
γ	= matrix of Lagrange multipliers
δ	= given flexibility matrix
Λ_y, Λ_w	= matrices of Lagrange multipliers
$\Lambda_{y,ij}, \Lambda_{w,ij}$	= ij element of Λ_y and Λ_w, respectively
Ω^2	= measured frequency matrix
Ω^2_{ij}	= ij element of Ω^2

I. Introduction

IN Ref. 1, the present author and Bar-Itzhack proposed a method by which a given stiffness matrix can be corrected optimally by using corrected mode shapes and natural frequencies obtained from vibration tests. However, Unger and Zalmanovitz[2] pointed out that the corrected stiffness matrix proposed in Ref. 1 does not fulfill a basic requirement, since the eigenvalues and the mode shape obtained from the solution of the dynamic equation using the corrected stiffness matrix usually are different from those used to obtain the corrected matrix. The main purpose of this Note is to propose a new corrected stiffness matrix, which is obtained following the approach applied in Ref. 1. However, instead of applying the insufficient contraint used in Ref. 1, a proper constraint is applied. The eigenvalues and the mode shapes obtained from the solution of the dynamic equation using the new corrected matrix will be the same ones used to correct the stiffness matrix.[3] For a large list of references, see Ref. 3.

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A similar method to correct optimally a given flexibililty matrix also is proposed.[4] The obtained flexibility matrix again incorporates the corrected measured mode shapes and the measured frequencies. In general, the solutions for the flexibility matrix and the stiffness matrix are different. However, for properly performed measurements and a good mathematical model of the structure, they cannot be too different. Therefore, the two solutions can be used to check the measurements and the mathematical model. The corrected stiffness or flexibility matrix can be used for further dynamic calculations. Hopefully, the additional mode shapes and their frequencies will be closer to the real (not measured) quantities.

II. Optimally Corrected Stiffness Matrix

In Ref. 1, a method was proposed by which the measured mode shapes obtained from vibration tests were corrected by minimization of a certain norm derived from the problem itself. The final result will be repeated here for the convenience of the reader:

$$X = T(T^t MT)^{-1/2} \tag{1}$$

where $X(n \times m)$ is the searched orthogonal mode shape matrix, $T(n \times m)$ is the measured mode shape matrix, and $M(n \times n)$ is the mass matrix. The mass matrix was assumed to be a known positive-definite symmetric matrix.

It must be noted that every measured mode T_i must be normalized in the following way:

$$T_i = \tilde{T}_i (\tilde{T}_i^t M \tilde{T}_i)^{-1/2} \tag{2}$$

where $\tilde{T}_i$ is the mode shape before normalization.

The mode shape matrix X obtained from Eq. (1) fulfills the basic orthogonal requirement

$$X^T MX = I \tag{3}$$

Note that X is a rectangular matrix of order $n \times m$, where $m \leq n$. Usually X does not contain all modes of the system.

Following Ref. 1, the most natural way to measure the closeness of two stiffness matrices Y and K is by evaluating the norm

$$g = \tfrac{1}{2} \| N^{-1}(Y-K)N^{-1} \| = \tfrac{1}{2} \bar{n}_{k\ell} (y_{\ell p} - k_{\ell p}) \bar{n}_{pq} \bar{n}_{kr} (y_{rs} - k_{rs}) \bar{n}_{sq} \tag{4}$$

where the Einstein rule of summation is applied, and $\bar{n}_{ij}$ are the elements of the matrix N^{-1}. $N(n \times n)$ is the positive-definite symmetric solution of the relation

$$N = M^{1/2} \tag{5}$$

Y is the searched corrected matrix, and K is a given stiffness matrix obtained by calculations or measurements. Usually K is obtained by using the finite-element method. K is a symmetric matrix that can be singular to include the rigid-body motions of the dynamic system.

In Ref. 1, it is shown that the norm [Eq. (4)] is consistent with the norm used to obtain Eq. (1). Now, as pointed out in Ref. 2, an insufficient constraint on the searched matrix was applied in Ref. 1. Here, a different constraint will be required from the matrix Y:

$$YX = MX\Omega^2 \tag{6}$$

where Ω^2 $(m \times m)$ is a diagonal matrix that represents the measured frequencies.

The constraint [Eq. (6)] is necessary and sufficient for X and Ω^2 to be solutions of the dynamic equation

$$M\ddot{q} + Yq = 0 \tag{7}$$

Multiplication of the left and right parts of Eq. (6) by the transpose of X and applying Eq. (3) yields

$$X^t YX = \Omega^2 \tag{8}$$

Equation (8) was used in Ref. 1 as a constraint on the searched stiffness matrix. As pointed out by Unger and Zalmanovitz,[2] this constraint is not sufficient for the X and Ω^2 to be solutions of Eq. (7) in the case when not all mode shapes and natural frequencies of the dynamic system are measured. In this case, which is the usual case, the X matrix is a rectangular matrix of order $(n \times m)$. Unger and Zalmanovitz[2] proposed a method in which the constraint [Eq. (8)] still is enforced but the X and Ω^2 matrices are modified.

Here a different approach will be applied. It is shown that Eq. (8) can be obtained from Eq. (6). The opposite is not true. For a rectangular matrix X $(n \times m)$, one cannot obtain Eq. (6) from Eq. (8). That is why constraint (8) is insufficient. However, constraint (6) includes in itself also Eq. (8).

The stiffness matrix Y must be symmetric. An additional requirement will be imposed on Y:

$$Y^t = Y \tag{9}$$

A mathematical formulation of the problem follows. Given is a rectangular matrix $X(n \times m)$, which represents the measured mode shapes of a given dynamic system and which satisfies the orthogonality conditions, Eq. (3). Given also is a diagonal matrix $\Omega^2 (m \times m)$, which represents the measured frequencies. Find a matrix Y that minimizes the weighted Euclidean norm [Eq. (4)] and satisfies constraints (6) and (9). Using Lagrange multipliers to incorporate the constraints, one obtains

$$G = g + 2\Lambda_{y,kp}(y_{k\ell}x_{\ell p} - m_{k\ell}x_{\ell q}\Omega^2_{qp}) + \beta_{y,k\ell}(y_{k\ell} - y_{\ell k}) \tag{10}$$

where Λ_y is a matrix of order $(n \times m)$, β_y an antisymmetric matrix of order $(n \times n)$,

$$\beta_y = -\beta_y^t \tag{11}$$

and

$$\Omega^2_{pq} = \Omega^2_p \quad \text{for} \quad p = q \tag{12}$$

$$\Omega^2_{pq} = 0 \quad \text{for} \quad p \neq q \tag{13}$$

where Ω_p are the measured frequencies.

The partial differentiation of G with respect to y_{ij} and equating the results to zero yields equations that y_{ij} have to satisfy when G is minimal:

$$\frac{\partial G}{\partial y_{ij}} = \bar{n}_{ki}\bar{n}_{jq}\bar{n}_{kr}(y_{rs} - k_{rs})\bar{n}_{sq} + 2\Lambda_{y,ip}x_{jp} + 2\beta_{y,ij}$$

$$= \bar{n}_{ik}\bar{n}_{kr}(y_{rs} - k_{rs})\bar{n}_{sq}\bar{n}_{qj} + 2\Lambda_{y,ip}x^t_{pj} + 2\beta_{y,ij} = 0 \tag{14}$$

Equation (14), written in matrix form, yields

$$\frac{\partial G}{\partial Y} = M^{-1}(Y - K)M^{-1} + 2\Lambda_y X^t + 2\beta_y = 0 \tag{15}$$

From Eq. (15), one obtains

$$Y = K - 2M\Lambda_y X^t M - 2M\beta_y M \tag{16}$$

Taking the transpose of Eq. (16), adding to Eq. (16), and taking into account Eqs. (9) and (11) yields

$$Y = K - M\Lambda_y X^t M - MX\Lambda_y^t M \tag{17}$$

By multiplication of Eq. (17) by X and taking into account Eqs. (6) and (3), one obtains

$$MX\Omega^2 = KX - M\Lambda_y - MX\Lambda_y^t MX \tag{18}$$

Now, a crucial assumption will be made. It will be assumed that the matrix $\Lambda_y^t MX$ is symmetric:

$$\Lambda_y^t MX = X^t M\Lambda_y \tag{19}$$

After obtaining the solution for Λ_y, one must check if it satisfies Eq. (19). Substitution of Eq. (19) into Eq. (18) yields

$$MX\Omega^2 = KX - M[I + XX^t M]\Lambda_y \tag{20}$$

It easily can be shown that

$$[I + XX^t M]^{-1} = [I - \tfrac{1}{2}XX^t M] \tag{21}$$

Using Eq. (21), one obtains from Eq. (20)

$$\Lambda_y = M^{-1}KX - \tfrac{1}{2}XX^t KX - \tfrac{1}{2}X\Omega^2 \tag{22}$$

It can be seen that Eq. (22) satisfies Eq. (19), and therefore assumption (19) is justified.

Substitution of Eq. (22) into Eq. (17) yields the corrected stiffness matrix:

$$Y = K - KXX^t M - MXX^t K + MXX^t KXX^t M + MX\Omega^2 X^t M \tag{23}$$

It easily can be shown that Eq. (23) satisfies Eq. (6). Hence, the mode shapes and eigenvalues obtained from the dynamic equation (7) will include the corrected mode shapes X and the measured frequencies Ω^2.

Now, it will be shown that the solution (23) is unique. Suppose that there exists a different solution γ for the Lagrange multiplier Λ_y and with it a different solution Y' for the stiffness matrix. From Eq. (17), one obtains

$$Y' = K - M\gamma X^t M - MX\gamma^t M \tag{24}$$

where Y' still has to satisfy Eq. (6). Multiplication of Eq. (24) by X and subtraction from Eq. (18) yields

$$M\Lambda_y + MX\Lambda_y^t MX = M\gamma + MX\gamma^t MX \tag{25}$$

Using Eq. (21), one obtains

$$\Lambda_y = \gamma + \tfrac{1}{2}X\gamma^t MX - \tfrac{1}{2}XX^t M\gamma \tag{26}$$

From Eq. (26), it is easy to obtain

$$M\gamma X^t M + MX\gamma^t M = M\Lambda_y X^t M + MX\Lambda_y^t M \tag{27}$$

Hence, whatever γ might be, substitution of Eq. (27) into Eq. (24) yields

$$Y' = Y \tag{28}$$

Equation (28) shows that the solution (23) is unique.

The total change of the Lagrange function G [Eq. (10)] is given by

$$\Delta G = \tfrac{1}{2}(\bar{n}_{k\ell}\Delta y_{\ell p}\bar{n}_{pq})^2 = \tfrac{1}{2}\|N^{-1}\Delta YN^{-1}\| \tag{29}$$

and is therefore always positive. Hence Y, obtained from Eq. (23), makes G [Eq. (10)] a minimum, and this is its only minimum.

It must be noted that, when X contains the full set of mode shapes, it is invertible, and then, using Eq. (3), it can be shown that

$$MXX^t = X^tMX = XX^tM = I \tag{30}$$

Substitution of Eq. (30) into Eq. (23) yields

$$Y_{\text{full}} = MX_{\text{full}}\Omega^2_{\text{full}}X^t_{\text{full}}M \tag{31}$$

III. Optimally Corrected Flexibility Matrix

Let us begin with the dynamic equation (7). Now it is supposed that the dynamic system is represented by a positive-definite symmetric flexibility matrix W. In this case, Eq. (7) becomes

$$WM\ddot{q} + q = 0 \tag{32}$$

To find the proper norm connected with the flexibility matrix, the following linear transformation will be applied:

$$p = Nq \tag{33}$$

Note that the same transformation was applied in Ref. 1 to obtain the proper norm for the modal matrix X and the stiffness matrix Y. Substitution of Eq. (33) in Eq. (32) yields

$$NWN\ddot{p} + p = 0 \tag{34}$$

It seems that the norm $\|NWN\|$ is the proper norm for the flexibility matrix W. W must be symmetric, hence,

$$W^t = W \tag{35}$$

The modal matrix $X(n \times m)$ and the measured frequencies matrix, $\Omega^2\ (m \times m)$, connected with it must solve Eq. (32):

$$WMX\Omega^2 = X \tag{36}$$

The closeness of two flexibility matrices W and δ will be measured by evaluation of the norm:

$$f = \tfrac{1}{2}\|N(W - \delta)N\| \tag{37}$$

where W is the searched corrected flexibility matrix, and δ is a given flexibility matrix obtained from calculations or experiment.

Now the formulation of the problem will be as follows. Find a matrix W that minimizes the norm [Eq. (37)] and that is subject to the constraints (35) and (36).

Using Lagrange multipliers to incorporate the constraints, one obtains

$$F = f + 2\Lambda_{w,kp}(w_{k\ell}m_{\ell q}x_{qr}\Omega^2_{rp} - x_{kp}) + \beta_{w,k\ell}(w_{k\ell} - w_{\ell k}) \tag{38}$$

Again, Λ_w is a matrix of order $(n \times m)$, and β_w is an antisymmetric matrix of order $(n \times n)$:

$$\beta^t_w = -\beta_w \tag{39}$$

The partial differentiation of F with rspect to w_{ij} and equating the results to zero yields equations thaat w_{ij} have to satisfy when F is minimal. Expressed in matrix form, this equation will be

$$\frac{\partial F}{\partial W} = M(W - \delta)M + 2\Lambda_w\Omega^2X^tM + 2\beta_w = 0 \tag{40}$$

Adding the transpose of Eq. (40) to Eq. (40) and considering Eq. (39) yields

$$W = \delta - M^{-1}\Lambda_w\Omega^2X^t - X\Omega^2\Lambda^t_wM^{-1} \tag{41}$$

By multiplication of Eq. (41) by $MX\Omega^2$ and taking into account Eqs. (36) and (3), one obtains

$$X = \delta MX\Omega^2 - M^{-1}\Lambda_w\Omega^4 - X\Omega^2\Lambda^t_wX\Omega^2 \tag{42}$$

Again, a crucial assumption will be made. It will be assumed that the matrix $\Omega^2\Lambda^tX$ is symmetric:

$$\Omega^2\Lambda^t_wX = X^t\Lambda_w\Omega^2 \tag{43}$$

Clearly, assumption (43) must be checked to see if it does not contradict the solution. Using Eq. (43), Eq. (42) becomes

$$[I + MXX^t]\Lambda_w\Omega^4 = M\delta MX\Omega^2 - MX \tag{44}$$

It can be verified that

$$[I + MXX^t]^{-1} = I - \tfrac{1}{2}MXX^t \tag{45}$$

and Eq. (44) can be solved for Λ_w to obtain

$$\Lambda^t_w = \Omega^{-2}X^tM\delta M - \tfrac{1}{2}\Omega^{-4}X^tM - \tfrac{1}{2}\Omega^{-2}X^tM\delta MXX^tM \tag{46}$$

It easily can be verified that Λ_w obtained from Eq. (46) satisfies Eq. (43). Substitution of Eq. (46) into Eq. (41) yields, finally,

$$W = \delta - \delta MXX^t - XX^tM\delta + XX^tM\delta MXX^t + X\Omega^{-2}X^t \tag{47}$$

As in Sec. II, it can be shown that Eq. (47) is a unique solution and is the only minimal point of the function F [Eq. (38)].

When X contains the full set of mode shapes, Eq. (47) becomes

$$W_{\text{full}} = X_{\text{full}}\Omega^{-2}_{\text{full}}X^t_{\text{full}} \tag{48}$$

Note that, when Y_{full} obtained from Eq. (31) is invertible, W_{full} is its inverse.

In general, W is not the inverse of Y. In other words, Eqs. (23) and (47) are, in general, two different solutions. Which one is better? This question can be answered only by experiments and experience. However, for basically sound measurements and computations, the two solutions cannot be too different. They can be used to check the measurements and the mathematical model used for the computations.

IV. Conclusions

Given stiffness and flexibility matrices are corrected optimally by using vibration tests. The two different solutions obtained can be used to check the measurements and the computations. The corrected matrices can be used for further dynamic computations.

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